Students revisit circles in the first part of Chapter 10 to develop “circle tools,” which will help them find lengths and angle measures within circles. In addition to working with the lengths of the radius and diameter of a circle, they will gain information about angles, arcs, and chords. As with the development of many topics we have studied, triangles will again be utilized.

See the Math Notes boxes in Lessons 10.1.1, 10.1.2, 10.1.3, 10.1.4, and 10.1.5.

Example 1

In the circle at right are two chords, $\overline{AB}$ and $\overline{CD}$. Find the center of the circle and label it $P$.

The chords of a circle (segments with endpoints on the circle) are useful segments. In particular, the diameter is a special chord that passes through the center. The perpendicular bisectors of the chords pass through the center of the circle as well. Therefore to find the center, we will find the perpendicular bisectors of each segment. They will meet at the center.

There are several ways to find the perpendicular bisectors of the segments. A quick way is to fold the paper so that the endpoints of the chords come together. The crease will be perpendicular to the chord and bisect it. Another method is to use the construction we learned last chapter. In either case, point $P$ is the center of the circle.
Example 2

In \( \odot O \) at right, use the given information to find the values of \( x, y, \) and \( z \).

Pieces of a circle are called arcs, and every arc breaks the circle into two pieces. The large piece is called a major arc, and the smaller piece is called a minor arc. Arcs have lengths, and we found lengths of arcs by finding a fraction of the circumference. But arcs also have measures based on the measure of the corresponding central angle. In the picture at right, \( \angle JOE \) is a central angle since its vertex is at the center, \( O \). An arc’s measure is the same as its central angle. Since \( JE = 100^\circ \), \( x = 100^\circ \).

An angle with its vertex on the circle is called an inscribed angle. Both of the angles \( y \) and \( z \) are inscribed angles. Inscribed angles measure half of their intercepted arc (in this case, \( JE \)). Therefore, \( y = z = \frac{1}{2} (100^\circ) = 50^\circ \).

Example 3

In the figure at right, \( O \) is the center of the circle. \( \overline{TX} \) and \( \overline{TB} \) are tangent to \( \odot O \), and \( m\angle BOX = 120^\circ \). Find the \( m\angle BTX \).

If a line is tangent to a circle, that line intersects the circle in only one point. Also, a radius drawn to the point of tangency is perpendicular to the tangent line. Therefore we know that \( \overline{OB} \perp \overline{BT} \) and \( \overline{OX} \perp \overline{XT} \). At this point there are different ways to solve this problem. One way is to add a segment to the picture. Adding \( \overline{OT} \) will create two triangles, and we know a lot of information about triangles. In fact, these two right triangles are congruent by \( \text{HL} \equiv ( \overline{OB} \equiv \overline{OX} \) because they are both radii, and \( \overline{OT} \equiv \overline{OT} \). Since the corresponding parts of congruent triangles are also congruent, and \( m\angle BOX = 120^\circ \), we know that \( m\angle BOT = m\angle XOT = 60^\circ \). Using the sum of the angles of a triangle is \( 180^\circ \), we find \( m\angle BTO = m\angle XTO = 30^\circ \). Therefore, \( m\angle BTX = 60^\circ \).

An alternate solution is to note that the two right angles at points \( B \) and \( X \), added to \( \angle BOX \), make \( 300^\circ \). Since we know that the angles in a quadrilateral sum to \( 360^\circ \), \( m\angle BTX = 360^\circ - 300^\circ = 60^\circ \).
Example 4

In the circle at right, $DV = 9$ units, $SV = 12$ units, and $AV = 4$ units. Find the length of $IV$.

Although we have been concentrating on angles and their measures, there are some facts about lengths of chords of circles that are useful (and should be part of your “circle tools”). In the figure above, if we drew $SI$ and $DA$ we would form two similar triangles. (See the Math Notes box in Lesson 10.1.4.) The sides of similar triangles are proportional, so we can write the proportion at right, which leads to the simplified equation with the two products.

$$\frac{SV}{DV} = \frac{IV}{AV}$$
$$SV \cdot AV = DV \cdot IV$$
$$\frac{SV}{DV} = \frac{IV}{AV}$$
$$\frac{12}{9} = \frac{IV}{4}$$
$$9IV = 48$$
$$IV = 5.33 \text{ units}$$

Problems

Find each measure in $\odot P$ if $m\angle WPX = 28^\circ$, $m\angle ZPY = 38^\circ$, and $WZ$ and $XV$ are diameters.

1. $YZ$
2. $WX$
3. $\angle VPZ$
4. $\overline{VWX}$
5. $\angle XPY$
6. $\overline{XY}$
7. $\overline{XWY}$
8. $\overline{WZX}$

In each of the following figures, $O$ is the center of the circle. Calculate the value of $x$ and justify your answer.

9.
10.
11.
12.
13.
14.
15.
16.
17.
18.
19.
20.
In $\odot O$, $m\overline{WT} = 86^\circ$ and $m\overline{EA} = 62^\circ$.

21. Find $m\angle EWA$.

22. Find $m\angle WET$.

23. Find $m\angle WES$.

24. Find $m\angle WST$.

In $\odot O$, $m\angle EWA = 36^\circ$ and $m\angle WST = 42^\circ$.

25. Find $m\angle WES$.

26. Find $m\overline{TW}$.

27. Find $m\overline{EA}$.

28. Find $m\angle TKE$.

29. In the figure at right, $m\overline{SD} = 92^\circ$, $m\overline{DA} = 103^\circ$, $m\overline{AI} = 41^\circ$ and $\overline{SW}$ is tangent to $\odot O$. Find $m\angle AKD$ and $m\angle VAS$.

30. In the figure at right, $m\overline{EK} = 43^\circ$, $\overline{EW} \equiv \overline{KW}$, and $\overline{ST}$ is tangent to $\odot O$. Find $m\angle WEO$ and $m\angle SEW$. 

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Core Connections Geometry
Answers

1. 38°  
2. 28°  
3. 28°  
4. 180°  
5. 114°  
6. 114°  
7. 246°  
8. 332°  
9. 68°  
10. 73°  
11. 98°  
12. 124°  
13. 50°  
14. 55°  
15. 18°  
16. 27°  
17. 55°  
18. 77°  
19. 35°  
20. 50°  
21. \( \frac{1}{2} (62°) = 31° \)  
22. \( \frac{1}{2} (86°) = 43° \)  
23. 180° – 43° = 137°  
24. 180° – 137° – 31° = 12°  
25. 180° – 36° – 42° = 102°  
26. \( m\angle TEW = 180° – 102° = 78°, 2(78°) = 156° \)  
27. 2(36°) = 72°  
28. 180° – 36° – 78° = 66°  
29. \( m\angle SAD = \frac{1}{2} (92°), m\angle IDA = \frac{1}{2} (41°), 180° – 46° – 20.5° = 113.5°, m\angle VAS = 180° – 46° = 134° \)  
30. \( m\angle EWK = \frac{1}{2} (43°) = 21.5°, m\angle EOK = 43°, \) so 317° remain for the other angle at O. \( m\angle WEO = m\angle WKO \) and for WEO, 360° – 21.5° – 317° = 21.5° = \( m\angle WEO + m\angle WKO \), so \( m\angle WEO = \frac{1}{2} (21.5°) = 10.75° \). \( m\angle SEO = 90°, m\angle WEO = 10.75° \), so \( m\angle SEW = 79.25° \).
The probability of one event occurring, knowing that a second event has already occurred is called a conditional probability. Two-way tables are useful to visualize conditional probability situations.

See the Math Notes boxes in Lessons 10.2.1 and 10.2.3.

Example 1

For the spinners at right, assume that the smaller sections of spinner #1 are half the size of the larger section and for spinner #2 assume that the smaller sections are one third the size of the larger section.

a. Draw a diagram for spinning twice.

b. What is the probability of getting the same color twice?

c. If you know you got the same color twice, what is the probability it was red?

The diagram for part (a) is shown at right. Note that the boxes do not need to be to scale. The circled boxes indicate getting the same color and the total probability for part (b) is: \( \frac{1}{8} + \frac{1}{12} + \frac{1}{24} = \frac{6}{24} = \frac{1}{4} \).

For part (c), both red is \( \frac{1}{8} \) out of the \( \frac{1}{4} \) from part (b) and so the probability the spinner was red knowing that you got the same color twice is \( \frac{1}{8} \div \frac{1}{4} = \frac{1}{8} \cdot 4 = \frac{1}{2} \).
Example 2

A soda company conducted a taste test for three different kinds of soda that it makes. It surveyed 200 people in each age group about their favorite flavor and the results are shown in the table below.

<table>
<thead>
<tr>
<th>Age</th>
<th>Soda A</th>
<th>Soda B</th>
<th>Soda C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under 20</td>
<td>30</td>
<td>44</td>
<td>126</td>
</tr>
<tr>
<td>20 to 39</td>
<td>67</td>
<td>75</td>
<td>58</td>
</tr>
<tr>
<td>40 to 59</td>
<td>88</td>
<td>78</td>
<td>34</td>
</tr>
<tr>
<td>60 and over</td>
<td>141</td>
<td>49</td>
<td>10</td>
</tr>
</tbody>
</table>

a. What is the probability that a participant chose Soda C or was under 20 years old?

b. What is the probability that Soda A was chosen?

c. If Soda A was chosen, what is the probability that the participant was 60 years old or older?

For part (a), using the addition rule:
\[
P(C \text{ or } <20) = P(C) + P(<20) - P(C \text{ and } <20) = \frac{238}{800} + \frac{200}{800} - \frac{126}{800} = \frac{312}{800} = 0.39.
\]

For part (b), adding the participants selecting Soda A:
\[
\frac{30+67+88+141}{800} = \frac{316}{800} = 0.395.
\]

For part (c), taking only the participants over 60 selecting Soda A out of all those selecting Soda A:
\[
\frac{141}{30+67+88+141} \approx 0.43
\]

Problems

1. Two normal dice are thrown.
   a. How many ways are there to get 7 points?
   b. What is the probability of getting 7 points?
   c. If you got 7 points, what is the probability that one die was a 5?

2. Elizabeth and Scott are playing game at the state fair that uses two spinners which are shown in the diagrams at right. The player spins both wheels and if the colors match you win a prize.
   a. Make a probability diagram for this situation
   b. What is the probability of winning a prize?
   c. If you won a prize, what is the probability that the matching colors were red?
3. The probability that it is Friday and a Sarah is absent is \( \frac{1}{30} \). Since the school week has 5 days, the probability it is Friday is \( \frac{1}{5} \). If today is Friday, what is the probability that Sarah is absent?

4. An airline wants to determine if passengers not checking luggage is related to people being on business trips. Data for 1000 random passengers at an airport was collected and summarized in the table below.

<table>
<thead>
<tr>
<th>Checked Baggage</th>
<th>No Checked Baggage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Traveling for business</td>
<td>103</td>
</tr>
<tr>
<td>Not traveling for business</td>
<td>216</td>
</tr>
</tbody>
</table>

a. What is the probability of traveling and not checking baggage?

b. If the passenger is traveling for business, what is the probability of not having checked baggage?

5. In Canada, 92% of the households have televisions. 72% of households have televisions and internet access. What is the probability that a house has internet given that it has a television?

6. There is a 25% chance that Claire will have to work tonight and cannot study for the big math test. If Claire studies, then she has an 80% chance of earning a good grade. If she does not study, she only have a 30% chance of earning a good grade.

a. Draw a diagram to represent this situation.

b. Calculate the probability of Claire earning a good grade on the math test.

c. If Claire earned a good grade, what is the probability that she studied?

7. A bag contains 4 blue marbles and 2 yellow marbles. Two marbles are randomly chosen.

a. What is the probability that both marbles are blue?

b. What is the probability that both marbles are yellow?

c. What is the probability of one blue and then one yellow?

d. If you were told that both marbles selected were the same color, what is the probability that both are blue?
8. At Cal’s Computer Warehouse, Cal wants to know the probability that a customer who comes into his store will buy a computer or a printer. He collected the following data during a recent week: 233 customers entered the store, 126 purchased computers, 44 purchased printers and 93 made no purchase.
   a. Draw a Venn diagram to represent the situation
   b. From this data, what is the probability that the next customer who comes into the store will buy a computer or a printer?
   c. Cal has promised a raise for his salespeople if they can increase the probability that the customers who buy computers also buy printers. For the given data, what is the probability that if a customer bought a computer, he or she also bought a printer?

9. A survey of 200 recent high school graduates found that 170 had driver licenses and 108 had jobs. Twenty-one said that they had neither a driver license nor a job.
   a. Draw a two-way table to represent the situation
   b. If one of these 200 graduates was randomly selected, what is the probability that he or she has a job and no license?
   c. If the randomly selected graduate is known to have a job, what is the probability that he or she has a license?

10. At McDougal’s Giant Hotdogs 15% of the part time workers are under 18 years old. The most desirable shift is 4-8pm and 80% of those under 18 have that shift. 30% of the other part time workers have that shift.
    a. Represent these probabilities in a two-way table.
    b. What is the probability that a randomly selected part time worker is 18 or over and does not work the 4-8pm shift?
    c. What is the probability that a randomly selected 18 year old or over part time worker does not work the 4-8pm shift
Answers

1. a. 6 ways  
   b. \( \frac{6}{36} = \frac{1}{6} \)  
   c. \( \frac{2}{36} + \frac{6}{36} = \frac{1}{3} \)

2. a. See diagram at right. 
   b. \( \frac{1}{6} + \frac{1}{12} = \frac{1}{4} \)  
   c. \( \frac{1}{6} + \frac{1}{4} = \frac{2}{3} \)

3. \( \frac{1}{20} + \frac{1}{5} = \frac{1}{4} \)

4. a. \( \frac{681}{1000} = 0.681 \)  
   b. \( \frac{387}{103+387} \approx 0.79 \)

5. \( \frac{72\%}{92\%} \approx 78\% \)

6. a. See diagram at right. 
   b. \( 0.075 + 0.60 = 0.675 = 67.5\% \)  
   c. \( \frac{0.60}{0.675} = 0.89 \)

7. a. \( \frac{4}{6} \cdot \frac{3}{5} = \frac{2}{5} \)  
   b. \( \frac{2}{6} \cdot \frac{1}{5} = \frac{1}{15} \)  
   c. \( \frac{4}{6} \cdot \frac{2}{5} = \frac{4}{15} \)  
   d. \( \frac{2}{3} + (\frac{2}{5} + \frac{1}{15}) = \frac{6}{7} \)

8. a. computer  printer  
   b. \( \frac{140}{233} \)
   c. \( \frac{41}{126} \)

9. a. See diagram at right. 
   b. \( \frac{9}{200} \)  
   c. \( \frac{99}{108} \)

10. a. See diagram at right. 
    b. \( 0.79 \)  
    c. \( \frac{0.79}{0.85} \approx 0.93 \)
PRINCIPLES OF COUNTING  10.3.1 –10.3.5

Students take on challenging problems using the Fundamental Principle of Counting, permutations, and combinations to compute probabilities. These techniques are essential when the sample space is too large to model or to count.

See the Math Notes boxes in Lessons 10.3.1, 10.3.2, 10.3.3, and 10.3.5.

Example 1

Twenty-three people have entered the pie-eating contest at the county fair. The first place pie-eater (the person eating the most pies in fifteen minutes) wins a pie each week for a year. Second place will receive new baking ware to make his/her own pies, and third place will receive the Sky High Pies recipe book. How many different possible top finishers are there?

Since the prizes are different for first, second, and third place, the order of the top finishers matters. We can use a decision chart to determine the number of ways we can have winners. How many different people can come in first? Twenty-three. Once first place is “chosen” (i.e., removed from the list of contenders) how many people are left to take second place? Twenty-two. This leaves twenty-one possible third place finishers. Just as with the branches on the tree diagram, multiply these numbers to determine the number of arrangements: 

$$ (23)(22)(21) = 10,626. $$

Example 2

Fifteen students are participating in a photo-shoot for a layout in the new journal Mathmaticious. In how many ways can you arrange:

a. Eight of them? b. Two of them? c. Fifteen of them?

We can use a decision chart for each of these situations, but there is another, more efficient method for answering these questions. An arrangement of items where order matters is called a **permutation**, and in this case, since changing the order of the students changes the layout, the order matters.

With a permutation, you need to know the total number things to be arranged (in this case \( n = 15 \) students) and how many will be taken ( \( r \) ) at a time. The formula for a permutation is

$$ nP_r = \frac{n!}{(n-r)!}. $$

In part (a), we have 15 students taken 8 at a time. The number of permutations is:

$$ 15P_8 = \frac{15!}{(15-8)!} = \frac{15!}{7!} = 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 259,459,200 $$
In part (b) the solution becomes: \(15P_2 = \frac{15!}{(15-2)!} = \frac{15!}{13!} = 15 \cdot 14 = 210\)

Part (c) poses a new problem: \(15P_{15} = \frac{15!}{(15-15)!} = \frac{15!}{0!}\)

What is 0!? “Factorial” means to calculate the product of the integers from the given value down to one. How can we compute 0!? If it equals zero, we have a problem because part (c) would not have an answer (dividing by zero is undefined). But this situation must have an answer. In fact, if we used a decision chart to determine how many ways the 15 people can line up, we would find that there are 15! arrangements. Therefore, if \(15P_{15} = 15!\) and 0! = 1. This is another case of mathematicians defining elements of mathematics to fit their needs. 0! is defined to equal 1 so that other mathematics makes sense.

**Example 3**

In the annual homecoming parade, three students get to ride on the lead float. Seven students are being considered for this coveted position. How many ways can three students be chosen for this honor?

All three students who are selected will ride on the lead float, but whether they are the first, second, or third student selected does not matter. In a case where the order of the selections does not matter, the situation is called a combination. This means that if the students were labeled A, B, C, D, E, and F, choosing A, B, and then C would be essentially the same as choosing B, C, and then A. In fact, all the arrangements of A, B, and C could be lumped together. This makes the number of combinations much smaller than the number of permutations. The symbol for a combination is \(\binom{n}{r}\) where \(n\) is the total number of items under consideration, and \(r\) is the number of items we will choose. It is often read as “\(n\) choose \(r\).” In this problem we have \(\binom{7}{3}\), 7 choose 3. The formula is similar to the formula for a permutation, but we must divide out the similar groups.

\[
nC_r = \frac{n!}{(n-r)!r!}
\]

Here we have: \(\binom{7}{3} = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3!} = 35\)
Problems

Simplify the following expressions.

1. \(10!\)  
2. \(\frac{10!}{3!}\)  
3. \(\frac{35!}{30!}\)  
4. \(\frac{88!}{87!}\)

5. \(\frac{72!}{70!}\)  
6. \(\frac{65!}{62!3!}\)  
7. \(8\ P_2\)  
8. \(15\ P_0\)

9. \(9\ P_9\)  
10. \(12\ C_4\)  
11. \(5\ C_0\)  
12. \(32\ C_32\)

Solve the following problems.

13. How many ways can you arrange the letters from the word “KAREN”?

14. How many ways can you arrange the letters from the word “KAREN” if you want the arrangement to begin with a vowel?

15. All standard license plates in Alaska start with three letters followed by three digits. If repetition is allowed, how many different license plates are there?

16. For $3.99, The Creamery Ice Cream Parlor will put any three different flavored scoops, out of their 25 flavors of ice cream, into a bowl. How many different “bowls” are there? (Note: A bowl of chocolate, strawberry, and vanilla is the same bowl as a bowl of chocolate, vanilla, and strawberry.)

17. Suppose those same three scoops of ice cream are on a cone. Now how many arrangements are there? (Note: Ice cream on a cone must be eaten “top down” because you cannot eat the bottom or middle scoop out, keeping the cone intact.)

18. A normal deck of playing cards contains 52 cards. How many five-card poker hands can be made?

19. How many ways are there to make a full house (three of one kind, two of another)?

20. What is the probability of getting a full house (three of one kind of card and two of another)? Assume a standard deck and no wild cards.

For problems 21–25, a bag contains 36 marbles. There are twelve blue marbles, eight red marbles, seven green marbles, five yellow marbles, and four white marbles. Without looking, you reach into the bag and pull out eight marbles. What is the probability you pull out:

21. All blue marbles?  
22. Four blue and four white marbles?

23. Seven green and one yellow marble?  
24. At least one red and at least two yellow?

25. No blue marbles?
Answers

1. 3,628,800
2. 604,800
3. 38,955,840
4. 88
5. 5,112
6. 43,680
7. 56
8. 1
9. 362,880
10. 495
11. 1
12. 1
13. 5! = 120
14. 2(4!) = 48
15. (26)(26)(26)(10)(10)(10) = 17,576,000
16. \(\binom{25}{3} = 2,300\)
17. \(25P_3 = 13,800\) (On a cone, order matters!)
18. \(\binom{52}{5} = 2,598,960\)
19. This is tricky and tough! There are 13 different “types” of cards: twos, threes, fours, ..., Jacks, Queens, Kings, and Aces. We need to choose which of the 13 we want three of \((\binom{13}{3})\). Once we choose what type (for example, we pick Jacks) then we need to choose which three out of the four to take \((\binom{4}{3})\). Then from the remaining 12 types, we choose which type to have two of \((\binom{12}{2})\). Then again we need to choose which two out of the four \((\binom{4}{2})\). This gives us \(\binom{13}{3} \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot \binom{4}{2} = 3,744\).
20. We already calculated the numbers we need in problems 18 and 19 so: \(\frac{3,744}{2,598,960} \approx 0.0014\).
21. Each time we reach in and pull out 8 marbles, order does not matter. The number of ways to do this is \(\binom{36}{8}\). This is the number in the sample space, i.e., the denominator. How many ways can we pull out all blue? \(\binom{12}{8}\). Therefore the probability is \(\frac{\binom{12}{8}}{\binom{36}{8}} \approx 0.0000164\).
22. Same denominator. Now we want to choose 4 from the 12 blue, \(\binom{12}{4}\), and 4 from the 4 whites, \(\binom{4}{4}\). \(\frac{\binom{12}{4} \cdot \binom{4}{4}}{\binom{36}{8}} \approx 0.0000164\), the same answer!
23. Seven green: \(7C_7\), one yellow: \(5C_1\). \(\frac{7C_7 \cdot 5C_1}{36C_8} \approx 0.0000001652\)
24. Here we have to get at least one red: \(8C_1\), and at least two yellow: \(5C_2\), but the other five marbles can come from the rest of the pot: \(33C_5\). Therefore, \(\frac{8C_1 \cdot 5C_2 \cdot 33C_5}{36C_8} \approx 0.627\).
25. To get no blue marbles means we want all eight from the other 24 non-blue marbles. \(\frac{24C_8}{36C_8} \approx 0.0243\).